Abstract
The behavior of rotating bodies such as gyroscopes and tippe tops was studied using a high speed camera in order to observe how applied forces affect their motion. First the motion of a gyroscope was examined with an external force applied. The gyroscope was found to precess, with a speed that could be quantitatively explained with simple equations using torque and angular momentum. The flipping of tippe tops was also studied. The top behavior could be qualitatively explained with simple equations. Friction and precession both played an important role in the motion of the tippe top. The tops appeared to stick at a fixed angle for some time before flipping. This was not predicted by the simplest model, but could be explained by the effect of gravity. Including this effect in the model would give better qualitative agreement between theory and experiment. It was found that the total kinetic plus potential energy of the tops was not conserved due to the frictional forces that cause the top to flip.

Introduction
Tops have fascinated children and adults for millennia due to their magical ability to balance unsupported for long periods of time. Many types of tops have been developed for a variety of purposes. Tops are often used in games, such as the ancient game of dreidel, in which a four-sided top is spun to win money or candy, that now has become a competitive sport. Children today battle with modern tops called “Bey Blades” that they crash into each other. The gyroscope is a device similar to a top that consists of a disk mounted on an axle that spins with high angular momentum. The gyroscope is set in a gimbaled mount. A gimbal consists of two rings interconnected by pivots so that an object can tilt in any direction. The gyroscope’s ability to maintain its orientation allows it to be used in navigation systems and aircraft instrumentation, as well as in the stabilization of devices such as torpedoes and boats.

Tops and gyroscopes exhibit motions under external forces that are known as precession and nutation. Precession is defined by Resnick, Halliday, and Crane as the motion in which “the axis of a rapidly spinning top will move slowly about the vertical axis”. Precession is a very nonintuitive behavior. If a gyroscope is spinning and a torque is applied perpendicular to its spin axis for a short period of time to tilt it, then it seems plausible that once the torque is no longer applied, the gyroscope will continue to tilt more and more. However, this cannot happen because then the direction of spin angular momentum would be changing, which is impossible because the torque has been removed. Instead, the gyroscope will tilt about the other axis. The explanation is as follows (Fig. 1a and b): Since the torque produced a change dL in angular momentum that is perpendicular to the gyroscope’s original angular momentum L, then once the torque stops, the gyroscope’s angular momentum has to point in the new direction given by L + dL. For a top with mass M, where the height of the center of mass is r, and the top is tilted angle θ away from vertical, the torque due to gravity is:

\[ N = \frac{dL}{dt} = Mr \sin(\theta) \]  

Equation 1.

The horizontal component of the top’s angular momentum is Lsinθ. The change in the direction of the top’s angular momentum is:

\[ \frac{d\phi}{dt} = \frac{Lr}{L} \]  

Equation 2.

The speed of precession is thus inversely proportional to the angular momentum L of the top, and therefore is also inversely proportional to its rotational speed.

Nutation is defined as “a slight oscillation of the axis of the top back and forth about the precessional circle”. There are many examples of precession and nutation in nature. The Earth’s rotational axis precesses, tracing out a cone in a 26,000 year cycle. The Earth nutates in its spinning path around the sun because of gravitational forces from the sun and moon, which are constantly changing location relative to each other.
This paper reports the study of two aspects of rotating bodies. First, a gyroscope was examined with an external force applied in order to observe precession. Next the motion of tippe tops was observed to see the effects of friction and gravity and to determine if energy was conserved. The tippe top is an excellent tool to analyze the physical behavior and forces that govern tops. It is shaped like a sphere with one section cut off that is replaced by an axle. The tippe top exhibits the fascinating action of rapidly turning over when it is spun with the curved side down, causing the heavy end to point upward as it is balanced on the spinning axle. This unexpected flipping is produced by the torque exerted through friction upon the top, causing the potential energy to increase at the expense of the rotational energy, which therefore decreases the angular momentum.4,6

Diagrams of the tippe top viewed from the side and from above are shown in Figure 2 (a) and (b) respectively. Since the tippe top consists mainly of a truncated sphere, its center of mass lies below the center of curvature of the sphere. When the top is tilted, the top touches the surface at a point A that lies below the center of curvature C and is offset from the axis of rotation about the center of mass CM. As a result, a torque N is exerted on the top due to the force of friction F with the surface. The center of mass is only offset a small amount from the center of curvature so the torque is almost horizontal. Since the top is spinning rapidly around the center of mass with frequency $\omega$, then the direction of this torque is also spinning at frequency $\omega$ and the average of the torque over time is zero. Therefore the direction of the angular momentum L of the top is conserved.4,6 In the reference frame of the body of the top, the torque is always perpendicular to the direction of tilt of the top. It therefore causes the angular momentum to precess in this frame so the top tilts more.4,6 The force due to friction is:

\[ F_f = \mu Mg \]  
Equation 3.

where $\mu$ is the coefficient of friction and g is the acceleration due to gravity. The torque due to friction is:

\[ N = F_f r \]  
Equation 4.

where r is the distance from the center of mass (cm) to the contact point at A. The precession or flip rate is therefore:

\[ \frac{d\theta}{dt} = \frac{N/L}{I\omega} = \frac{(\mu Mg)r}{(1/2)r^2\omega} \]  
Equation 5.

$I$ is the moment of inertia of the top and $\omega$ is the rotation frequency.

Since the center of mass is very close to the center of curvature, then r is approximately equal to the diameter of the body of the tippe top. If the moment of inertia of the tippe top is approximated by that of a hollow sphere:

\[ I = 2/3(Mr^2) \]  
Equation 6.

then the flip rate becomes:

\[ \frac{d\theta}{dt} = \frac{N/L}{I\omega} = \frac{(\mu Mg)r}{(3/2)\omega} \]  
Equation 7.

This equation reveals how the parameters of the tippe top affect its flip rate. The flip rate does not depend on the mass of the tippe top. The flip rate increases when the coefficient of friction $\mu$ increases because the torque due to friction increases. The flip rate decreases when the spin rate $\omega$ increases because the angular momentum L becomes larger so it is harder for the torque to flip the top. Also the flip rate increases when the top radius r decreases because the angular momentum of the top is smaller so it is easier to flip.
Materials and Methods

The gimbaled gyroscope shown in Figure 3(a) was used to study precession. It consisted of a rotor on an axle supported by bearings. The axle was mounted in a gimbaled ring that was free to tilt. The gimbaled ring was supported by a bracket which rested on a base that allowed it to pivot freely about the vertical axis. The rotor was covered with a piece of paper with one half colored black to make it easier to observe the rotation of the rotor with high speed video. To apply an external torque, paperclips were clipped to the ring of the gimbaled gyroscope. To start the gyroscope, a string was inserted into a hole in the shaft and the rotor was rotated to wind up the string on the shaft. The axle was oriented horizontally. The string was then pulled vigorously in order to spin the rotor at high speed. The initial speed depended on how hard the string was pulled. Video of the spinning rotor was then recorded using a Casio EX-FH-100 camera capable of recording 120, 240, 420, or 1000 frames/second. Normal playback speed for this video was 30 frames/second so motion was slowed down by 4, 8, 14, or 33 1/3 times. Video of the gyroscope was recorded with either two paperclips or four paperclips attached. The videos analyzed here were recorded with a speed of 420 frames/second. Normal playback occurred at 30 frames/second so this corresponded to slowing down the motion by 14 times. Videos were played back using the open source VLC media player software. This software allowed slowing down or speeding up the playback in order to better see the rotation. First, the gyroscope rotation rate as a function of time was determined. At intervals in the video of 140 seconds, corresponding to every 10 seconds real time, 100 revolutions of the gyroscope rotor were counted and the start and finish times were recorded. The times were then divided by 1400 to find the real time for a single revolution of the gyroscope rotor and then converted to either revolutions per minute (RPM) or radians/sec. Next, the precession rate of the gyroscope was measured. The time for each quarter turn of the gyroscope was determined by recording the times at which the rotor axis pointed directly at the camera and the times at which the gimbal axis pointed at the camera. In order to calculate theoretical values of the precession for comparison with this experiment, the dimensions and masses of the gyroscope rotor and the locations and weights of the paperclips were precisely measured. The gyroscope was disassembled and the rotor was weighed with an American Weigh SC-2kg scale and found to weigh 142.3 g. The rotor dimensions were measured with digital calipers. The rotor measurements are shown in Figure 4. In order to calculate the torque produced by the paperclips, their distances relative to the pivot axis of the gimbals were measured. As shown in Figure 5, two of the paperclips were at distance 3 cm from the gimbaled axis and two paperclips were at distance 4.1 cm from the gimbal axis. The paper clips weighed approximately 2.4 g. The tippe top was spun on a table and then the motion was recorded with the high speed camera. The camera lens was held level with the plane of the table so that the tippe top could be observed from the side, making it possible to record the angle of tilt. Measurements were performed using 420 frames/sec corresponding to a slowdown of 14 times. Half of the top was colored black with a marker in order to make it easier to see the rotation. Trials were performed using a plastic top, a wooden top, and a metal top. The dimensions and masses of the three tippe tops differed. The dimensions of the tops were measured with calipers and they were weighed on an American Weigh SC-2kg scale (Table 1). In this experiment, the tippe tops flipped and the measurements were over in five seconds or less. The VLC software proved inadequate to observe the motion, therefore the VirtualDub open source video capture/processing software was used. This video editing program allowed playing back videos frame by frame, making it easier to count short times. In order to measure the rotation rate, the number of frames for the top to make five revolutions was counted. Initially the top was spinning about its axis so that transitions from uncolored to colored portion could be counted. As the top tilted its angular momentum remained vertical so that once the top was horizontal, no rotation occurred about the axis of its shaft. The revolutions of the shaft about the vertical direction were then counted. The number of frames for five revolutions was divided by five and then by 420 frames/sec to get the time for one revolution. This was converted to rotation rate in radians/sec and rpm. Two trials of the plastic tippe top and one trial each of the wooden
and metal tippe tops were performed. In order to measure the angle from vertical of the tippe tops over time, the video was moved to each time at which the velocity was previously measured. The video was then stepped until the tip pointed as far as possible to the left or right. To measure the deviation from vertical a pencil was held parallel with the shaft and then an on-screen image of a protractor was used to measure the angle of the pencil from vertical. The video was next advanced so the tip pointed the opposite way and the angle measured again.

Table 1. Dimensions of tops and calculated moments of inertia. The formula used is for a hollow sphere of diameter \( a^4 \).

<table>
<thead>
<tr>
<th></th>
<th>Plastic Top</th>
<th>Wooden Top</th>
<th>Metal Top</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of sphere (cm)</td>
<td>3.52</td>
<td>3.17</td>
<td>2.89</td>
</tr>
<tr>
<td>Radius of Sphere (cm)</td>
<td>1.76</td>
<td>1.585</td>
<td>1.445</td>
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<tr>
<td>Height of truncated sphere (cm)</td>
<td>2.81</td>
<td>2.76</td>
<td>2.31</td>
</tr>
<tr>
<td>Height of top including post (cm)</td>
<td>4.09</td>
<td>3.93</td>
<td>3.55</td>
</tr>
<tr>
<td>Distance from center of sphere to end of post (cm)</td>
<td>2.33</td>
<td>2.34</td>
<td>2.10</td>
</tr>
<tr>
<td>Length of post (cm)</td>
<td>1.28</td>
<td>1.16</td>
<td>1.24</td>
</tr>
<tr>
<td>Extra Height of post over radius of sphere (cm)</td>
<td>0.57</td>
<td>0.76</td>
<td>0.65</td>
</tr>
<tr>
<td>Mass (grams)</td>
<td>6.6</td>
<td>9.41</td>
<td>18.95</td>
</tr>
<tr>
<td>Moment of Inertia, ( I = \frac{2}{3}Ma^2 ) (kg m(^2))</td>
<td>(1.36 \times 10^{-6})</td>
<td>(1.57 \times 10^{-6})</td>
<td>(2.65 \times 10^{-6})</td>
</tr>
</tbody>
</table>

Results

Gyroscope

The rotation direction of the rotor is shown in Figure 3a. When a torque was applied to the gyroscope by the gravitational force on the paperclips, it caused the gyroscope to precess.\(^4\)

Figure 3b illustrates the principle of precession. In time \( dt \), the torque \( N \) applied by the paperclips produced a change \( dL \) in the angular momentum \( L \) which was at right angles to the original angular momentum \( L \) of the spinning gyroscope rotor. This caused the gyroscope to precess through an angle \( \phi \). The rate of precession can therefore be written as\(^4\):

\[
\omega_p = \frac{d\phi}{dt} = \frac{N}{L}
\]

Equation 8.

Since \( L = I\omega \)

Equation 9.

then Equation 8 can be written as:

\[
\omega_p = \frac{N}{I\omega}
\]

Equation 10.

Note that as \( \omega \) decreases \( \omega_p \) increases.

Figure 6 shows the gyroscope spin rate versus time for one trial with two paperclips attached to the gimbal and a second trial with four paperclips attached. Rotation rates as high as 3,600 rpm or 60 revolutions/second were observed. In both trials the rotation rate decelerated with time.

The diamonds and the triangles in Figure 7 show the precession rates measured with either four paperclips or two paperclips applying torque to the gimbaled ring. The precession rate generally increased with time as the gyroscope slowed down and its angular momentum decreased.
Tippe Top
The rotational velocity versus time is plotted in Figure 8 for the three different tippe tops. The vertical dashed lines show the times at which the posts touched the table. The rates were as high as 2,600 rpm, greater than 40 revolutions/sec or 10 frames/revolution.

The angular velocity decreased roughly linearly. When the axle touched the table the velocity decreased much faster. For the plastic top this occurred at 3 seconds for trial 1 and 2.1 seconds for trial 2. For the wooden top the tip did not touch until 6.8 seconds. The slope of speed of the metal top was steeper and its tip touched after 2.3 seconds.

Figure 9 shows the top angles versus time for the different tops. The vertical dashed lines represent the times at which the posts touched the table and the tops began to rise. The wooden top which slowed down the most gradually, took the longest time to flip. The plastic top, which slowed down faster, took less time to flip. Trial 2 of the plastic top started at a lower spin rate than trial 1, but flipped faster than trial 1. The metal top flipped the fastest. Once the posts of the tops touched down and the tops began to rise then the rate of angle change increased. Also, for the plastic and wooden tops the angle reached a plateau at around 20 – 40 degrees at which it stayed for a long time and then eventually started to tip more rapidly. The metal top did not have a plateau, but tipped slowly at first and then accelerated.

In order to see if the total energy was conserved both the kinetic and potential energy of the top were calculated. The kinetic energy was given by:

\[ E = \frac{1}{2} I \omega^2 \]  

\[ \text{Equation 11.} \]

\( \omega \) is the rotational velocity of the top, converted to units of radians/second and \( I \) is the moment of inertia. In order to calculate the moment of inertia of each top it was assumed that the tops were hollow spheres. The moment of inertia is given by Equation 6:\n
The actual moment of inertia was smaller since the material was actually not all at the surface. Also, the tops were not completely spherical so the moment of inertia would have different values for different directions and therefore change as the top tilts. The calculated values are shown in Table 1.

To calculate the potential energy the initial height of the center of mass was taken to be 0 and it was assumed that the center of mass was at the center of the sphere. The height only increases after the post touches the table top. From Figure 10 the height is given by:

\[ H = D \cos(\pi - \theta) - r \]  

\[ \text{Equation 12.} \]
The kinetic energy, potential energy and total energy are plotted in Figure 11 a – d for the four trials. The potential energy was calculated from the top dimensions and masses in Table 1, the angles measured in Figure 9 and the heights calculated from Equation 12. The kinetic energies were calculated from the moments of inertia in Table 1 and the velocities plotted in Figure 8.

**Figure 8.** Tippe top rotation velocity versus time. The black and maroon curves are for the two trials of the plastic top. The yellow and light blue curves are for the wooden and metal tops. The vertical dashed lines show where the posts touched the table. The rotation speeds decrease almost linearly until the posts touch the table, at which point the rates decrease faster. The tops that decelerate faster also flip faster.

**Discussion**

**Gyroscope**

The total torque exerted on the gyroscope by the paperclips was calculated using their masses and distances from the gimbal pivots that were previously measured. For four paperclips the torque was calculated to be:

\[ N = \sum M g X_i = 2 \times 2.4 \times 10^{-3} \text{ kg} \times 9.8 \text{ m/s}^2 \times (0.031 \text{ m} + 0.041 \text{ m}) = 3.34 \times 10^{-3} \text{ kg m}^2/\text{s}^2 \]

Here \( X_i \) was the distance of the ith paperclip from the gimbaled axis.

Equation 13.

Similarly, the total torque exerted by the two paperclips was:

\[ N = \sum M g X_i = 2 \times 2.4 \times 10^{-3} \text{ kg} \times 9.8 \text{ m/s}^2 \times 0.041 \text{ m} = 1.93 \times 10^{-3} \text{ kg m}^2/\text{s}^2 \]

Equation 14.

Note that most of the mass of the gyroscope rotor shown in Figure 4 was in the outer ring. The disc-like area between the rotor and the shaft was thin and had holes drilled in it, so it did not have much mass. The axle was narrow so it did not weigh much either. The moment of inertia, \( I \), of the rotor was therefore calculated by assuming that the mass was all concentrated in the outer ring at a distance equal to the average of the inner and outer dimensions of the ring.

Average radius \( a = (5.89 \text{ cm} + 4.88 \text{ cm})/4 \)
\( a = 2.6925 \text{ cm} \times 1 \text{ m/100 cm} = 0.026925 \text{ m} \)
\( M = 0.1433 \text{ kg} \)
\( I = Ma^2 = 1.0316 \times 10^{-4} \text{ kg m}^2 \)

The angular momentum of the gyroscope was calculated using Equation 9, where the rotation speed \( \omega \) was taken from the values plotted in Figure 6 and converted to radians/sec. The theoretical precession rate was calculated using Equation 8. This is shown in Figure 7 as the light blue curve for four paperclips and the pink curve for two paperclips. The theoretical curve fits the data quite well for the four

**Figure 9.** Tippe top angle versus time. 0 degrees is when the top is upright. 180 degrees is when the top is upside down. Tops were spun counterclockwise when viewed from the top.

**Figure 10.** The height of the tippe top above the table.

\[ c - \text{Center of sphere} \]
\[ D - \text{Distance from center of sphere to end of post} \]
\[ r - \text{Radius of sphere} \]
\[ \theta - \text{Angle post makes with vertical} \]
\[ H - \text{Distance from sphere to table} \]
Figure 11. Kinetic energy, potential energy, and total energy for (a) trial 1 and (b) trial 2 of the plastic top, (c) the wooden top, and (d) the metal top. The total energy is constantly decreasing, so energy is not conserved. The maximum increase in potential energy is much smaller than the decrease in kinetic energy for all tops.

The decrease in velocities of the tops over time as seen in Figure 8 was due to friction. The slope of the speed of the wooden top was several times less than that of the plastic top. This means that the friction was a few times less. The explanation for the faster deceleration of the tops seen in Figure 8 once the posts touched down to the table is as follows: When the top was rotating on its curved part, the offset paperclip case. The theoretical curve is slightly below the experimental curve. This may be because the assumption that all the gyroscope mass was concentrated in the rotor ring overestimated the moment of inertia and therefore the angular momentum $L$ of the rotor. A more exact calculation of the moment of inertia of the gyroscope rotor would decrease the calculated values of the angular momentum $L$ and therefore increase the calculated precession rate $\omega_p$ slightly.

For the two paperclip case some measured points jumped above and below the theoretical curve and also the precession rate dropped at the end. The drop at the end occurred because by that time the gimbaled ring was no longer horizontal so the actual torque due to the paperclips was much less than was estimated.

The variation in precession rate of the two paperclip case may be due to friction of the gimbal pivots that reduced the torque. When pivoting the gyroscope the gimbals sometimes seemed to stick a little bit. A small spring scale was used to measure the force needed to make the unweighted gimbaled ring pivot. It was found that it could take a force of as much as 1.5 grams to tilt it, which is almost as much as one paperclip. The force to rotate the gyroscope horizontally was also measured and found to be negligible. At times when the gimbals stuck the precession rate would decrease, causing the variation that was seen.

Tippe Top
The decrease in velocities of the tops over time as seen in Figure 8 was due to friction. The slope of the speed of the wooden top was several times less than that of the plastic top. This means that the friction was a few times less. The explanation for the faster deceleration of the tops seen in Figure 8 once the posts touched down to the table is as follows: When the top was rotating on its curved part, the offset
of the contact from the center of rotation was small, so the energy loss \( F_i \cdot V \) due to friction was small. However, when the tip touched the table, its contact point was at a large distance from the axis of rotation and it was therefore moving at high velocity. The friction energy loss therefore became large. As the top straightened up, the radius of the movement of the tip decreased, so when the top finished flipping, the velocity became relatively constant again.

The total energy of each top plotted in Figure 11 was constantly decreasing while the top was tipping, so energy was not conserved. After the post touched and the top started to rise, the total energy decreased even faster. This is because the end of the post was moving at high velocity so the friction energy loss \( F_i \cdot V \) was large. The maximum increase in potential energy was much smaller than any decrease in kinetic energy. Once the tops had completely flipped then since the post was very narrow the energy loss \( F_i \cdot V \) due to friction was small and the tops only lost energy slowly.

For the plastic and wooden tops the reason that the angle of the post from vertical shown in Figure 9 stayed constant for a long time may be that the center of mass was below the center of the sphere, so it did take some force to tilt the top away from vertical. This can be seen since a tilted tippe top that is not spinning will always come to rest with the post pointing straight up. Once the top is tilted enough to overcome this force, it tips more rapidly. The plateau in the first plastic top trial was more apparent than in the second plastic top trial because it took longer to flip and therefore longer to overcome the force of gravity.

Table 2 shows the flip rates calculated for the tops using Equation 7. The coefficients of friction for plastic and metal on glass were taken to be 0.5. Since the wooden top slowed down much more slowly, a lower coefficient of friction of 0.2 was used for calculating its flip rate. The predicted flip times of the tops were somewhat shorter than what was measured but qualitatively explain the relative lengths of the measured times. The second plastic top trial flipped more quickly than the first since the spin speed and therefore angular momentum were lower. The wooden top flipped more slowly since the coefficient of friction seemed lower. The longer measured times were likely again caused by the tops remaining at a constant tilt for a while due to the effect of gravity.

Possible errors were that the coefficients of friction were not well known since they depended on the cleanliness of the surface. Also, Equation 7 was derived assuming that the moment of inertia was that of a hollow sphere, but actually the tippe tops were asymmetric and the center of mass was not exactly at the center of the sphere. When measuring the top angles with the post pointing first left and then right it was often seen that these angles differed from each other. The pictures were blurry so this could be from measurement error, or, it could be that the tops were nutating. There could also be some error if the camera was not completely vertical. Since the final angles were close to 180 degrees the camera was not too far off from vertical however.

The precession of a gyroscope with an external force applied has been studied experimentally. It was found that its behavior could be explained with a simple equation using torque. Next, as a more complicated example of precession, the flipping of tippe tops made of different materials when spun at different speeds was observed. The tippe top behavior was qualitatively explained with simple equations. Precession played an important role in the behavior of the tippe top since the torque due to friction caused the top to precess in the reference frame rotating with it, making it tip. It was found that friction seemed to be important for causing the tipping, since tops that lost speed more slowly and hence were less affected by friction with the table also flipped more slowly. Additionally, the energy loss due to friction increased when the post touched the table and the top rose up to vertical. Energy was not conserved during the tipping. It was also observed that the tops appeared to stick at a fixed angle for a time before going on to tip. This was believed to be due to the effect of gravity since a tilted top that is not spinning will fall back to the position with the post sticking straight up. The angle between the top and vertical sometimes varied within a half a turn. This may be a sign of nutation since nutation often appears as a bobbing motion when a top precesses.
For the future, it would be interesting to observe nutation in the behavior of the gyroscopes and tippe tops. Nutation is caused when an additional angular momentum is added perpendicular to the axis of rotation of the top causing it to appear to bob as it precesses. The direction of the gyroscope exhibits cycloidal looking motion. A gyroscope can be attached to a counterbalanced arm (shown in Figure 12). This gyroscope system exhibits strong nutation when perturbed while precessing. The end of the gyroscope shows a fascinating cycloidal looking motion when observed on the slow motion video.

It would also be useful to measure the coefficients of friction between the tippe tops and the table in order to have accurate values to calculate the torques due to friction. Euler derived differential equations of motion for a rotating body in a coordinate system moving with the rotating body. Solving these equations of motion would permit modeling the motion of the tippe tops more exactly using the correct moments of inertia.

References


